The concept of Tyranny of the Majority is centred around the idea that the majority opinion does
not always reflect what is best for society. This is due to the different weights that people assign
to different decisions and outcomes. The value that I may get out of decision X is likely to be
different than the value another person gets out of decision X. One person may even receive a
negative value from the decision and be worse off.

Tyranny of the Majority occurs when the majority of the population benefits from a decision that
makes the entire population collectively worse off. For example, take a group of ten people where
six people benefit from a decision by $10, and four people suffer from the same decision by $30.
If the decision comes down to a vote, the six people who benefit would vote in favour while the
other four who suffer would vote against. Since the decision passes by majority vote, the six
people who benefit collectively gain $60, while the four people who suffer collectively lose $120.
Therefore, even though the majority got their way, the group collectively looses $60.

We will use Harsanyi's Additive Welfare Function to find a model for Tyranny of the Majority.
Harsanyi's Additive Welfare Function can be written as the following:

$$W(p_k, s_k) = \sum_{i=1}^{N} EU_i(p_k, s_k), \text{ where } EU_i(p_k, s_k) = \sum_{k=1}^{K} p_k u_i(s_k)$$

where $s_k$ represents one outcome in the set of $K$ outcomes, $p_k$ represents the probability of $s_k$
occurring, $i$ denotes an individual, $u_i$ is a utility function that represents how much value the $i$th
person receives out of $s_k$, $EU_i$ represents the expected utility the $i$th individual receives out of the
set that contains every $s_k$, and $W$ is the societal welfare function which represents the collective
expected utility of every $i$ person.

In order to come up with a model for Tyranny of the Majority, we will divide up the welfare function into two parts: the welfare of the majority and the welfare of the minority. Let $A$ represent the majority of a group of $N$ people. Mathematically, this means that the following conditions are on $A$:

$$A > \frac{N}{2} \text{ and } N > A$$

The welfare of the majority is therefore defined as:

$$W^+(p_k, s_k) = \sum_{i=1}^{A} EU_i(p_k, s_k)$$

The welfare of the minority is simply the difference between the societal welfare and the welfare of the majority. The welfare of the minority is therefore:

$$W^-(p_k, s_k) = W(p_k, s_k) - W^+(p_k, s_k) = \sum_{i=1}^{N} EU_i(p_k, s_k) - \sum_{i=1}^{A} EU_i(p_k, s_k)$$

$$= \sum_{i=A+1}^{N} EU_i(p_k, s_k)$$

To apply this to the Tyranny of the Majority, we will make the following behavioural assumptions:

1. A person will vote in favour of a decision when they receive an increase of benefits.

   o $EU_i(p_k, s_k) > 0$

2. A person will vote against a decision when they receive a decrease of benefits.

   o $EU_i(p_k, s_k) < 0$

3. Society collectively benefits from a decision when the societal welfare increases.

   o $W(p_k, s_k) > 0$

4. Society collectively suffers from a decision when the societal welfare increases.

○ $W(p_k, s_k) < 0$

Without loss of generality, assume that the majority group $A$ is in favour of a decision. This means that everybody in $A$ will receive an increase of benefits. Mathematically, this means:

$$EU_1, \dots, EU_A > 0$$

Likewise, the minority are against the same decision. This means that the rest of the group will incur a decrease in benefits. Mathematically, this means:

$$EU_{A+1}, \dots, EU_N < 0$$

Given these assumptions, we now know the following properties of the welfare functions for the majority and minority groups:

$$W^+(p_k, s_k) = \sum_{i=1}^{A} EU_i(p_k, s_k) > 0$$

$$W^-(p_k, s_k) = \sum_{i=A+1}^{N} EU_i(p_k, s_k) < 0$$

Tyranny of the Majority occurs when the majority vote does not make society better off. In this case, the societal welfare function will be less than zero. To see what that looks like, we input the welfare of the majority and welfare of the minority variables into assumption 3 from above:

$$W^+(p_k, s_k) + W^-(p_k, s_k) < 0 \quad \rightarrow \quad -W^-(p_k, s_k) > W^+(p_k, s_k)$$

Therefore, in a population of $N$ people and $A$ people make up the majority, Tyranny of the Majority happens when the population votes on a decision with the following utility relationship:

$$-\sum_{i=A+1}^{N} EU_i(p_k, s_k) > \sum_{i=1}^{A} EU_i(p_k, s_k)$$

In other words, Tyranny of the Majority happens when the majority votes for a decision that will make them better off, but the collective society worse off. Note that in the case that the majority $A$ votes against the decision, then following the same logic would yield a welfare relationship of:

$$W^-(p_k, s_k) > -W^+(p_k, s_k)$$

Aside from schoolwork, one thing I spend a lot of time doing is governance. I am on the board of directors of Waterloo Co-operative Residence Inc. (WCRI), a student-housing co-operative in Waterloo. It is the job of myself and my fellow directors to set the strategic vision for the organization to make sure WCRI is what the membership wants it to be. One of the ways we do this is by setting policies that govern the co-operative and make decisions on behalf of the membership. But how do we determine which policies and decisions benefit the co-operative, and which ones do more harm than good?

Let us look at an example of building a study room in one of our buildings. I will first define the following variables and make the following assumptions:

- $N$ is the entire population of WCRI.
- $B$ is the population of the building that will have the study room.
  - Since this building is one of many buildings, $B$ is therefore the minority population.
- $C$ is the amount every member will need to pitch in in order to build the study room.
  - Note that $C < 0$.
- $S$ is the benefit that one individual in the population of $B$ gets out of the study room.
- Only the people who live in the building with the study room receive its benefits.

From the statements above, we can model the following conclusions:

- The welfare of the people in the building (the minority) is positive, and can be represented by:

$$W^- = \sum_{i=1}^{B} (S + C) = B(S + C) = BS + BC$$

- The welfare of the people not in the building (the majority) is negative, and can be represented by:

$$W^+ = \sum_{i=B+1}^{N} C = C(N - B) = NC - BC$$

If this decision came to a vote, then the majority would vote against the construction of the study room and it would not be built. However, this might not be the best outcome for the collective membership. As stated above, when the majority of the population opts to vote against a decision, Tyranny of the Majority happens when:

$$W^- > -W^+$$

In words, this means that WCRI would have been collectively better off with the construction of the study room despite the construction coming at a cost to the majority.

To solve for the conditions in which Tyranny of the Majority occurs, we will substitute our values for welfare of the majority and welfare of the minority into the above inequality:

$$BS + BC > -(NC - BC)$$

$$BS + BC > BC - NC \quad \rightarrow \quad \frac{B}{N} > \frac{-C}{S}$$

Note that since $C < 0$, $-C/S$ is a positive ratio.

This is a very simple, yet very important result. This result means that when the minority group benefits from a project at the cost of the majority, the society as a whole benefits when the ratio between the minority population and the entire population is greater than the ratio between the cost of the project per person in the population and the benefits of the project per person in the minority.

With the way that WCRI's governance is structured, decisions such as whether to build a study room or not do not come to a vote but are rather decisions made by the board or staff. This way, in the case that the minority/population ratio is greater than the cost/benefit ratio, we avoid the "tyrannical" vote where the majority would vote against the study room, and instead the decision is made based on what is better for WCRI.

While I derived this result in the context of WCRI, it is simple to generalize the idea to any case where one wants to determine whether the society benefits at the cost to the majority. If one wanted to determine the case where the majority votes for something in their favour at the cost of the minority, they would get the following result ($A$ represents the majority of the population):

$$\frac{-C}{S} > \frac{A}{N}$$

meaning that society is not better off when the vote passes if the above inequality is true. The tricky part becomes determining the costs and benefits to the respective groups, as the costs and benefits people incur from decisions are not always a direct monetary value. In the example of the study room, most of the benefit comes from students being able to study more easily and peacefully. This is not a benefit that has an easily quantifiable value, and people like our board and staff need to find alternative ways to measure this value.